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INVARIANT DIFFERENTIAL OPERATORS ON A REAL SEMISIMPLE LIE ALGEBRA AND THEIR RADIAL COMPONENTS

BY

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ABSTRACT. Let $S(\mathfrak{g}_c)$ be the symmetric algebra over the complexification $\mathfrak{g}_c$ of the real semisimple Lie algebra $\mathfrak{g}$. For $u \in S(\mathfrak{g}_c)$, $\partial(u)$ is the corresponding differential operator on $\mathfrak{g}$. $\mathfrak{D}(\mathfrak{g})$ denotes the algebra generated by $\partial(S(\mathfrak{g}_c))$ and multiplication by polynomials on $\mathfrak{g}_c$. For any open set $U \subset \mathfrak{g}$, $\mathrm{Diff}(U)$ is the algebra of differential operators with C^∞-coefficients on U. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$, $\mathfrak{h}'$ the set of its regular points and $\pi = \Pi_{\alpha \in P}\, \alpha$, P some positive system of roots. Let $W = (\mathfrak{h}')^G$, G the connected adjoint group of $\mathfrak{g}$.

Harish-Chandra showed that, for each $D \in \mathrm{Diff}(W)$, there is a unique differential operator $\delta_{\mathfrak{h}}'(D)$ on $\mathfrak{h}'$ such that $(Df)|_{\mathfrak{h}'} = \delta_{\mathfrak{h}}'(D)(f|_{\mathfrak{h}})$ for all G-invariant $f \in C^\infty(W)$, and that if $D \in \mathfrak{D}(\mathfrak{h})$, then $\delta_{\mathfrak{h}}'(D) = \pi^{-1} \circ \overline{D} \circ \pi$ for some $\overline{D} \in \mathfrak{D}(\mathfrak{g})$. In particular $\overline{\partial(u)} = \partial(u|_{\mathfrak{h}})$, $u \in S(\mathfrak{g}_c)$ and invariant.

We prove these results by different, yet simpler methods. We reduce evaluation of $\delta_{\mathfrak{h}}'(\partial(u))$ ($u \in S(\mathfrak{g}_c)$, invariant) via Weyl's unitarian trick, to the case of compact G. This case is proved using an evaluation of a family of G-invariant eigenfunctions on:

$$\pi(H)\pi(H')\int_G \exp B(H^x, H')\,dx = c \sum_{s \in W(\mathfrak{g}_c, \mathfrak{h}_c)} \epsilon(s)\exp B(sH, H'), \qquad H, H' \in \mathfrak{g},\ c > 0.$$

For G-invariant $D \in \mathfrak{D}(\mathfrak{g})$, we prove $\pi^{-1} \circ \delta'(D) \circ \pi \in \mathfrak{D}(\mathfrak{h})$ using properties of derivations $E \to [\partial(u), E]$ of $\mathfrak{D}(\mathfrak{g})$ induced by $\partial(u)$ ($u \in S(\mathfrak{g}_c)$) and of the algebra of polynomials on $\mathfrak{h}_c$ invariant under the Weyl group.

1. **Preliminaries.** Our aim here is to give alternative proofs to some results of Harish-Chandra on radial components of invariant differential operators on a real semisimple Lie algebra [1], [2].

Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbf{R}$, $\mathfrak{g}_c$ its complexification, $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$, and $\mathfrak{h}_c$ complexification of $\mathfrak{h}$. Denote by $\mathfrak{g}'$ the set of regular elements of $\mathfrak{g}$ and $\mathfrak{h}' = \mathfrak{h} \cap \mathfrak{g}'$. We have $H \in \mathfrak{h}'$ if and only if $\pi(H) \neq 0$,

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where $\pi = \Pi_{\alpha \in P}\, \alpha$, P being a positive system of roots of $\mathfrak{g}_c$ with respect to $\mathfrak{h}_c$. Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. Let $S(\mathfrak{g}_c)$ and $P(\mathfrak{g}_c)$ be the symmetric and the polynomial algebras over $\mathfrak{g}_c$, and $Q(\mathfrak{g}_c)$ quotient field of $P(\mathfrak{g}_c)$. We assume as usual that $\mathfrak{g}_c \subset S(\mathfrak{g}_c)$. Let U be an open subset of $\mathfrak{g}$, and $C^\infty(U)$ the algebra of C^∞ functions on U. An element $X \in \mathfrak{g}$ acts as a derivation of $C^\infty(U)$ given by

$$(Xf)(u) = df(u + tX)/dt\big|_{t=0} \tag{1}$$

where $f \in C^\infty(U)$, $u \in U$, and $t \in \mathbf{R}$. We denote the above derivation by $\partial(X)$ and extend the map $\partial\colon X \mapsto \partial(X)$ uniquely to a homomorphism of $S(\mathfrak{g}_c)$ into the associative algebra of endomorphisms of $C^\infty(U)$. If $\phi \in C^\infty(U)$, then ϕ is identified with the endomorphism $\phi\colon f \mapsto \phi f$ of $C^\infty(U)$. The algebra generated by $\{\phi, \partial(p) \mid \phi \in C^\infty(U),\ p \in S(\mathfrak{g}_c)\}$ is denoted by $\operatorname{Diff}(U)$. It is called the algebra of differential operators on $\mathfrak{g}$ with C^∞ coefficients. We denote by $\mathfrak{D}(\mathfrak{g})$ the algebra generated by $P(\mathfrak{g}_c) \cup \partial(S(\mathfrak{g}_c))$, and refer to it as the algebra of differential operators on $\mathfrak{g}$ with polynomial coefficients. We write $f(u; D)$ to mean $(Df)(u)$, for $D \in \operatorname{Diff}(U)$, $u \in U$, $f \in C^\infty(U)$. For any $D \in \operatorname{Diff}(U)$, and $u \in U$, there is a unique $p \in S(\mathfrak{g}_c)$, such that $f(u; D) = f(u; \partial(p))$. $\partial(p)$ is called the local expression of D at u, and is denoted by D_u.

Let U be an open subset of $\mathfrak{g}$ invariant under G. G acts naturally on $C^\infty(U)$ and $\operatorname{Diff}(U)$ if we set

$$f^x(X) = f(X^{x^{-1}}) \qquad (x \in G,\ X \in U,\ f \in C^\infty(U))$$

and

$$D^x f = (Df^x)^{x^{-1}} \qquad (D \in \operatorname{Diff}(U),\quad x \in G,\quad f \in C^\infty(U)).$$

D (resp. f) is called invariant under G if $D^x = D$ (resp. $f^x = f$) for all $x \in G$. Let $I^\infty(U) = \{f\colon f \in C^\infty(U),\ f^x = f \text{ for all } x \in G\}$, and $\operatorname{Diff}_{\mathrm{inv}}(U) = \{D \in \operatorname{Diff}(U)\colon D^x = D \text{ for all } x \in G\}$. We put $\mathfrak{I}(\mathfrak{g}) = \{D \in \mathfrak{D}(\mathfrak{g}) \mid D^x = D \text{ for all } x \in G\}$.

We now take $U = (\mathfrak{h}')^G$. Then, corresponding to any $D \in \operatorname{Diff}_{\mathrm{inv}}(U)$, Harish-Chandra [1] constructed a differential operator $\delta'_{\mathfrak{h}}(D)$ on $\mathfrak{h}'$ such that

$$f(H; D) = \bar{f}(H; \delta'_{\mathfrak{h}}(D)) \qquad (\bar{f} = f|_{\mathfrak{h}'}) \tag{2}$$

for all $f \in I^\infty(U)$, and $H \in \mathfrak{h}'$. $\delta'_{\mathfrak{h}}(D)$ is called the *radial component* of D on $\mathfrak{h}'$ and we have the following result (Harish-Chandra [1]):

Theorem 1.1. *Given any $D \in \operatorname{Diff}_{\mathrm{inv}}((\mathfrak{h}')^G)$, there is exactly one $\delta'_{\mathfrak{h}}(D) \in \operatorname{Diff}(\mathfrak{h}')$ such that*

$$f(H; D) = \bar{f}(H; \delta'_{\mathfrak{h}}(D)) \qquad (f \in I^\infty((\mathfrak{h}')^G)).$$

The operator $\delta'_{\mathfrak{h}}(D)$ is invariant under the normalizer of $\mathfrak{h}$ in G. The map $D \longrightarrow \delta'_{\mathfrak{h}}(D)$ is a homomorphism of $\operatorname{Diff}_{\mathrm{inv}}((\mathfrak{h}')^G)$ into $\operatorname{Diff}(\mathfrak{h}')$.

It turns out to be important to calculate $\delta'_{\mathfrak{h}}(D)$ explicitly for at least the most important of the invariant differential operators D. This was first done by Harish-Chandra. We shall now describe two of his results. Let $I_s(\mathfrak{g}_c)$ be the algebra of G-invariant elements of $S(\mathfrak{g}_c)$, and $I_s(\mathfrak{h}_c)$ the subalgebra of $S(\mathfrak{h}_c)$ of elements invariant under the Weyl group $W(\mathfrak{g}_c, \mathfrak{h}_c)$. If $p \in I_s(\mathfrak{g}_c)$, then by a theorem of Chevalley, there is a unique $p_{\mathfrak{h}} \in I_s(\mathfrak{h}_c)$ such that $p - p_{\mathfrak{h}}$ is in the ideal generated by the root-spaces. Harish-Chandra [1] showed that, for $p \in S(\mathfrak{g}_c)$,

$$\delta'_{\mathfrak{h}}(\partial(p)) = \pi^{-1} \circ \partial(p_{\mathfrak{h}}) \circ \pi. \tag{3}$$

Generalizing this, he proved in [2] that, for any $D \in \mathfrak{D}(\mathfrak{g}_c)$ which is invariant, $\pi \circ \delta'_{\mathfrak{h}}(D) \circ \pi^{-1}$ is the restriction to $\mathfrak{h}'$ of an element of $\mathfrak{D}(\mathfrak{h}_c)$ that is invariant with respect to $W(\mathfrak{g}_c, \mathfrak{h}_c)$.

We shall obtain these two results by a method that is somewhat different from Harish-Chandra's. We shall prove (3) first when G is compact and then extend it for noncompact G by the "unitarian trick". The second result is then deduced from formula (3). The proof of (3) for compact G is quite simple and goes as follows. By invariant integration and the Weyl character formula, we obtain explicit formulae for a class of invariant eigenfunctions on $\mathfrak{g}$, the formula (3) applied to these eigenfunctions now determines $\delta'_{\mathfrak{h}}(\partial(p))$ uniquely.

2. **Case when** G **is compact.** We now assume that G is compact and simply connected. Let B be the analytic subgroup of G corresponding to $\mathfrak{h}$. B is a maximal torus of G and $G = B^G$. Exp is a homomorphism of $\mathfrak{h}$ onto B. Let $\hat{B}$ be the character group of B. If $\xi \in \hat{B}$, $\xi \circ \exp$ is a character of $\mathfrak{h}$, and so there is a $\mathbf{C}$-linear function λ on $\mathfrak{h}_c$, which takes pure imaginary values on $\mathfrak{h}$, such that $\xi \circ \exp = e^{\lambda}$. λ is uniquely determined by ξ, and we write $\xi = \xi_{\lambda}$. Thus $\xi_{\lambda}(\exp H) = e^{\lambda(H)}$ $(H \in \mathfrak{h})$.

Let $\mathfrak{L}$ be the set of all linear functions on $\mathfrak{h}_c$ with the property that $e^{\lambda} = \xi \circ \exp$ for some $\xi \in \hat{B}$. $\mathfrak{L}$ is an additive subgroup of $\mathfrak{h}_c^*$ and the correspondence $\lambda \longrightarrow \xi_{\lambda}$ $(\lambda \in \mathfrak{L})$ is an isomorphism of $\mathfrak{L}$ onto $\hat{B}$. Since G is simply connected, $\mathfrak{L}$ is precisely the set of all integral linear functions on $\mathfrak{h}_c$ [6]. If α is a root of $(\mathfrak{g}_c, \mathfrak{h}_c)$, then $\alpha \in \mathfrak{L}$. Now B is the centralizer of $\mathfrak{h}$ in G, $\mathrm{Ad}(B)$ leaves the root-subspaces corresponding to α invariant, and we have $X_{\alpha}^{b} = \xi_{\alpha}(b) X_{\alpha}$ $(b \in B$, $X_{\alpha} \in \mathfrak{g}_{\alpha})$. Define $\delta = \frac{1}{2} \sum_{\alpha \epsilon P} \alpha$; δ is integral, and so $\delta \in \mathfrak{L}$. Let $\Delta = \xi_{-\delta} \prod_{\alpha \epsilon P} (\xi_{\alpha} - 1)$. Δ is a finite sum of characters of B. If we write $\epsilon(s) = \det(s)$ for $s \in W = W(\mathfrak{g}_c, \mathfrak{h}_c)$, it is known that $\Delta = \sum_{s \in W} \epsilon(s) \xi_{s\alpha}$. Denote by B' the set of $b \in B$ such that $\Delta(b) \neq 0$, i.e. $\xi_{\alpha}(b) \neq 1$ for all $\alpha \in P$.

Let $\mathfrak{D}_p^+$ be the set of all dominant integral linear functions on $\mathfrak{h}_c$. For any $\lambda \in \mathfrak{D}_p^+$, let π_{λ} be the irreducible representation of $\mathfrak{g}_c$ with highest weight λ. Since G is simply connected, π_{λ} lifts to a representation of G, denoted by π_{λ} again.

Let $\psi_\lambda(x) = \operatorname{tr} \pi_\lambda(x)$, $x \in G$. We then have the following formula of Weyl.

$$\psi_\lambda(b) = \sum_{s \in W} \epsilon(s) \xi_{s(\lambda+\delta)}(b) \Big/ \Delta(b) \qquad (b \in B'). \tag{4}$$

In particular, for $H \in \mathfrak{h}$ for which $\exp H \in B'$,

$$\operatorname{tr} \pi_\lambda(\exp H) = \sum_{s \in W} \epsilon(s) e^{s(\lambda+\delta)(H)} \Big/ \prod_{\alpha \in P} (e^{\alpha(H)/2} - e^{-\alpha(H)/2}) \tag{5}$$

and

$$\dim \pi_\lambda = \prod_{\alpha \in P} \langle \lambda + \delta, \alpha \rangle \Big/ \prod_{\alpha \in P} \langle \delta, \alpha \rangle. \tag{6}$$

Let $\mathfrak{G}$ be the universal enveloping algebra of $\mathfrak{g}_c$ and $\mathfrak{Z}$ the center of $\mathfrak{G}$. By Schurr's lemma there is a homomorphism χ_λ (called the infinitesimal character of π_λ) of $\mathfrak{Z}$ into $\mathbf{C}$, such that $\pi_\lambda(z) = \chi_\lambda(z) \cdot 1$ for all $z \in \mathfrak{Z}$. Let $\mathfrak{H}$ be the subalgebra of $\mathfrak{G}$ generated by 1 and $\mathfrak{h}_c$, and let $\mathfrak{P}$ be the left ideal $\Sigma_{\alpha \in P} \mathfrak{G} X_\alpha$. Then it can be shown that for each $z \in \mathfrak{Z}$, there is a unique element $\beta_p(z) \in \mathfrak{H}$ such that $z - \beta_p(z) \in \mathfrak{P}$. The map β_p: $z \longrightarrow \beta_p(z)$ is a homomorphism of $\mathfrak{Z}$ onto $\mathfrak{H}$, and (cf. Harish-Chandra [4])

$$\chi_\lambda(z) = \beta_p(z)(\lambda) \qquad (z \in \mathfrak{Z}). \tag{7}$$

Let U be a G-invariant open subset of $\mathfrak{g}$ and $f \in C^\infty(U)$. We say f is an invariant eigenfunction on $\mathfrak{g}$, if f is G-invariant, and for each $p \in I_s(\mathfrak{g}_c)$, there is a $\chi(p) \in \mathbf{C}$ such that

$$\partial(p) f = \chi(p) f \quad \text{on } U.$$

χ is uniquely determined if $f \neq 0$ and $\chi(p \longrightarrow \chi(p))$ is a homomorphism of $I_s(\mathfrak{g}_c)$ into $\mathbf{C}$.

Lemma 2.1. *Let dx be the normalized Haar measure on G, and $X' \in \mathfrak{g}_c$. Define*

$$f_{X'}(X) = f(X' : X) = \int_G \exp B(X^x, X')\, dx \qquad (X \in \mathfrak{g}).$$

Then $f_{X'}$ is an invariant eigenfunction on $\mathfrak{g}$, and

$$\partial(p) f_{X'} = \tilde{p}(X') f_{X'} \qquad (p \in I_s(\mathfrak{g}_c))$$

where $p \longrightarrow \tilde{p}$ is the isomorphism of $S(\mathfrak{g}_c)$ onto $P(\mathfrak{g}_c)$ induced by the Cartan-Killing form.

Proof. Let $\phi(x, X) = \exp B(X^x, X')$, $x \in G$, $X \in \mathfrak{g}_c$. Then $\phi \in C^\infty(G \times \mathfrak{g})$ and as G is compact, the integral defined by $f_{X'}(X)$ exists. The function $X \longrightarrow f_{X'}(X)$ is of class C^∞ on $\mathfrak{g}$ and we can differentiate under the integral sign. If $p \in S(\mathfrak{g}_c)$,

$$f_{X'}(X; \partial(p)) = \int_G \phi(x : X; \partial(p))\, dx \qquad (X \in \mathfrak{g}).$$

Let $\omega \in \mathfrak{g}_c^*$. Then using the natural identification of $S(\mathfrak{g}_c)$ with $P(\mathfrak{g}_c^*)$, we obtain $\partial(p)e^\omega = p(\omega)e^\omega$. If $Y \in \mathfrak{g}_c$ and ω is the linear function $X \longrightarrow B(X, Y)$, then $p(\omega) = \tilde{p}(Y)$. From these and the fact that $\phi(x, X) = \exp B(X, X'^{x^{-1}})$, we get, for all $x \in G$, $X \in \mathfrak{g}$,

$$\phi(x, X; \partial(p)) = \tilde{p}(X'^{x^{-1}})\phi(x, X).$$

If we assume $p \in I_s(\mathfrak{g}_c)$, we obtain $\partial(p)f_{X'} = \tilde{p}(X')f_{X'}$. Invariance of $f_{X'}$ follows from translation invariance of dx.

Our aim in the rest of this section is to evaluate the $f_{X'}$. This requires some preparation. For any subspace $\mathfrak{a}$ of $\mathfrak{g}_c$, we shall identify $S(\mathfrak{a})$ with the subalgebra of $S(\mathfrak{g}_c)$ generated by 1 and $\mathfrak{a}$. For any $n \geq 0$, write $S_n(\mathfrak{g}_c)$ for the subspace of $S(\mathfrak{g}_c)$ spanned by homogeneous elements of degree n. Set $S^{(n)}(\mathfrak{g}_c) = \Sigma_{0 \leq r \leq n} S_r(\mathfrak{g}_c)$. Let $\tilde{\lambda}$ be the customary symmetrizer map of $S(\mathfrak{g}_c)$ onto $\mathfrak{G}$. Since $\tilde{\lambda}(p^x) = \tilde{\lambda}(p)^x$ $(p \in S(\mathfrak{g}_c), x \in G)$, $\tilde{\lambda}$ is a linear bijection of $I_s(\mathfrak{g}_c)$ onto $\mathfrak{Z}$. It is also known to be a linear bijection of $S^{(n)}(\mathfrak{g}_c)$ onto the subspace $\mathfrak{G}^{(n)}$ of $\mathfrak{G}$ spanned by 1 and elements of the form $X_1 \cdots X_r$ $(1 \leq r \leq n,\ X_i \in \mathfrak{g}_c)$.

Lemma 2.2. *Let $\mathfrak{L}$ be the subspace of $\mathfrak{g}_c$ spanned by X_α, $\alpha \in \Delta$. Then $S(\mathfrak{g}_c) = S(\mathfrak{h}_c) + S(\mathfrak{g}_c)\mathfrak{L}$. For any $p \in S(\mathfrak{g}_c)$, let $p_\mathfrak{h}$ denote the unique element of $S(\mathfrak{h}_c)$ such that $p - p_\mathfrak{h} \in S(\mathfrak{g}_c)\mathfrak{L}$. Then $p \longrightarrow p_\mathfrak{h}$ is an isomorphism of $I_s(\mathfrak{g}_c)$ onto $I_s(\mathfrak{h}_c)$. If $p \in I_s(\mathfrak{g}_c) \cap S_n(\mathfrak{g}_c)$ then $p_\mathfrak{h} \in I_s(\mathfrak{h}_c) \cap S_n(\mathfrak{h}_c)$.*

Proof. The statements are all consequences of the theorem of Chevalley mentioned earlier.

Put $\overline{\mathfrak{H}} = \{a \in \mathfrak{H}: a^s = a \text{ for all } s \in W\}$. The map $\tilde{\lambda}$, by restriction, induces an isomorphism of $S(\mathfrak{h}_c)$ onto $\mathfrak{H}$, and $I_s(\mathfrak{h}_c)$ onto $\overline{\mathfrak{H}}$. Let $\mathfrak{H}_n = \tilde{\lambda}(S_n(\mathfrak{h}_c))$. $\mathfrak{H}$ is the direct sum of $\mathfrak{H}_n$'s, and for any $v \in \mathfrak{H}$, we write v_n for its component in $\mathfrak{H}_n$. Put $\mathfrak{H}^{(n)} = \Sigma_{0 \leq r \leq n} \mathfrak{H}_r$. Then

Lemma 2.3. *Let $p \in I_s(\mathfrak{g}_c) \cap S_n(\mathfrak{g}_c)$. Then $\tilde{\lambda}(p) \in \mathfrak{Z} \cap \mathfrak{G}^{(n)}$, $\beta_p(\tilde{\lambda}(p)) \in \mathfrak{H}^{(n)}$, $\beta_p(\tilde{\lambda}(p))_n = \tilde{\lambda}(P_\mathfrak{h})$.*

Proof. Let $\{H_1, \cdots, H_l\}$ be a basis for $\mathfrak{h}_c$ and $P = \{\alpha_1, \cdots, \alpha_d\}$. Let $(q) = (q_1, \cdots, q_d)$, $(c) = (c_1, \cdots, c_l)$, $(r) = (r_1, \cdots, r_d)$, where q_i, c_j, and r_k are integers ≥ 0. Define

$$M((q), (c), (r)) = X_{-\alpha_1}^{q_1} \cdots X_{-\alpha_d}^{q_d} \cdot H_1^{c_1} \cdots H_l^{c_l} \cdot X_{\alpha_1}^{r_1} \cdots X_{\alpha_d}^{r_d}.$$

$p \in I_s(\mathfrak{g}_c)$ if and only if it is a linear combination of $M((q), (c), (r))$ for which $\Sigma_{i=1}^d (r_i - q_i)\alpha_i = 0$. Hence we can find constants $A((q), (c), (r))$ (all but finitely many of them zero) such that

$$p = p_{\mathfrak{h}} + \sum_{\Omega} A((q), (c), (r))M((q), (c), (r)) \tag{8}$$

where the sum is over the set Ω of all $((q), (c), (r))$ for which $\Sigma_{i=1}^d (r_i - q_i)\alpha_i = 0$, $\Sigma_i r_i > 0$, $\Sigma_i q_i > 0$ and $\Sigma_{i=1}^d (r_i + q_i) = n$.

On the other hand by considering $M((q), (c), (r))$ as an element of $\mathcal{G}$, we can show that if $z \in \mathcal{Z} \cap \mathcal{G}^{(n)}$, then $\beta_p(z) \in \mathcal{H}^{(n)}$. Let $z = \tilde{\lambda}(p)$, then using (8) and the result that $\tilde{\lambda}(M((q),(c),(r))) \in \mathcal{P} + \mathcal{G}^{(n-1)}$, and the fact that β_p maps $\mathcal{G}^{(n-1)}$ into $\mathcal{H}^{(n-1)}$ we conclude

$$\beta_p(z) \in \tilde{\lambda}(p_{\mathfrak{h}}) + \mathcal{H}^{(n-1)}.$$

But $\tilde{\lambda}(p_{\mathfrak{h}}) \in \mathcal{H}_n$. Hence (9) implies the result.

Consider now $u \in S_n(\mathfrak{g}_c)$, then $p = \int_G u^x\,dx$ is a well-defined element of $I_s(\mathfrak{g}_c) \cap S_n(\mathfrak{g}_c)$. Also forming $\int_G \tilde{\lambda}(u)^x\,dx$, we obtain an element z of $\mathcal{Z} \cap \mathcal{G}^{(n)}$. Since $\tilde{\lambda}(u^x) = \tilde{\lambda}(u)^x$, $x \in G$, we have $z = \tilde{\lambda}(p)$. Lemma 2.3, now shows that $\beta_p(z)_n = \tilde{\lambda}(p_{\mathfrak{h}})$. Taking $u = H^n$ for $H \in \mathfrak{h}_c$ we get

Lemma 2.4. *Let $n \geq 0$. Then for any $H \in \mathfrak{h}_c$*

$$\zeta_{H,n} = \int_G (H^n)^x\,dx$$

is a well-defined element of $I_s(\mathfrak{g}_c) \cap S_n(\mathfrak{g}_c)$ while considering the same integral in $\mathcal{G}$, we obtain an element $z_{H,n} \in \mathcal{Z} \cap \mathcal{G}^{(n)}$. We have $\tilde{\lambda}(\zeta_{H,n}) = z_{H,n}$ and $\beta_p(z_{H,n})_n = \tilde{\lambda}(\zeta_{H,n}|_{\mathfrak{h}})$.

Corollary 2.5. *Let $\mu \longrightarrow H'_\mu$ be the isomorphism of $\mathfrak{h}_c^*$ with $\mathfrak{h}_c$ induced by the Cartan-Killing form. Then for any $H \in \mathfrak{h}_c$, $n \geq 0$ and $\mu \in \mathfrak{h}_c^*$,*

$$\int_G B(H^x, H'_\mu)^n\,dx = \beta_p(z_{H,n})(\mu). \tag{9}$$

Proof. From Lemma 2.4, we have $\beta_p(z_{H,n})(\mu) = \tilde{\lambda}(\zeta_{H,n}|_{\mathfrak{h}})(\mu)$ for $\mu \in \mathfrak{h}_c^*$. From isomorphism $p \longrightarrow \tilde{p}$ of $S(\mathfrak{g}_c)$ with $P(\mathfrak{g}_c)$ and the definition of $\zeta_{H,n}$, we obtain

$$\zeta_{H,n}(Y) = \int_G B(H^x, Y)^n\,dx \qquad (H \in \mathfrak{h},\ Y \in \mathfrak{g}_c).$$

So for $\mu \in \mathfrak{h}_c^*$

$$\int_G B(H^x, H'_\mu)\,dx = \tilde{\zeta}_{H,n}(H'_\mu).$$

On the other hand, the map $p \longrightarrow p_{\mathfrak{h}}$ corresponds, via the isomorphism $p \longrightarrow \tilde{p}$, to the restriction map $\tilde{p} \longrightarrow \tilde{p}|_{\mathfrak{h}_c}$ of $P(\mathfrak{g}_c)$ onto $P(\mathfrak{h}_c)$ so $\tilde{\zeta}_{H,n}|_{\mathfrak{h}} = (\zeta_{H,n}|_{\mathfrak{h}})^{\sim}$, from which we get $\tilde{\zeta}_{H,n}(H'_\mu) = (\zeta_{H,n}|_{\mathfrak{h}})(\mu)$, $\mu \in \mathfrak{h}_c^*$. Since $\tilde{\lambda}(\bar{H}^n) = \bar{H}^n$, $\bar{H} \in \mathfrak{h}_c$, $\tilde{\lambda}(\zeta_{H,n}|_{\mathfrak{h}_c})(\mu) = (\zeta_{H,n}|_{\mathfrak{h}_c})(\mu)$.

We are now in a position to obtain a formula for f_χ, defined in Lemma 2.1.

Theorem 2.6. *Let G be a compact and semisimple real Lie group, $\mathfrak{g}$ its Lie algebra, $\mathfrak{h} \subset \mathfrak{g}$ a Cartan subalgebra. P a positive system of roots, $\pi = \Pi_{\alpha\in P}\alpha$, $\tilde{\omega} = \Pi_{\alpha\in p}H'_\alpha$. Then for all $H, H' \in \mathfrak{h}$ we have*

$$\pi(H)\pi(H')\int_G \exp B(H^x, H')\,dx = \left(\prod_{\alpha\in P}\langle\delta, \alpha\rangle\right)\sum_{s\in W}\epsilon(s)e^{B(sH,H')}.$$

Proof. We have $\exp B(X, Y) = \Sigma_{n\geq 0} B(X, Y)^n/n!$. Since $|B(X^x, Y)|$ is bounded as x varies in G, we can write

$$\int_G \exp B(H^x, H')\,dx = \sum_{n\geq 0}\frac{1}{n!}\int_G B(H^x, H')^n\,dx.$$

Upon replacing H' by H'_μ ($\mu \in \mathfrak{h}_c^*$) in above and using Corollary 2.5, we get

$$\int_G \exp B(H^x, H'_\mu)\,dx = \sum_{n\geq 0}\frac{1}{n!}\beta_p(z_{H,n})(\mu).$$

For $\lambda \in \mathfrak{D}_p^+$, let π be the irreducible representation of $\mathfrak{g}_c$ and G with highest weight λ. From (7) we obtain

$$\beta_p(z_{H,n})(\lambda) = \chi_\lambda(z_{H,n}).$$

Hence, as $z_{H,n} \in \mathfrak{Z}$

$$\operatorname{tr}\pi_\lambda(z_{H,n}) = (\dim \pi_\lambda)\chi_\lambda(z_{H,n}) = (\dim \pi_\lambda)\beta_p(z_{H,n})(\lambda).$$

Using (6) in above we get

$$\tilde{\omega}(\lambda + \delta)\beta_p(z_{H,n})(\lambda) = \prod_{\alpha\in p}\langle\delta, \alpha\rangle \operatorname{tr}\pi_\lambda(z_{H,n}). \tag{10}$$

But from the definition of $z_{H,n}$,

$$\operatorname{tr}\pi_\lambda(z_{H,n}) = \int_G \operatorname{tr}\pi_\lambda(H^n)^x\,dx = \operatorname{tr}\pi_\lambda(H^n).$$

Now, for $t \in \mathbf{R}$,

$$\pi_\lambda(\exp tH) = e^{t\pi_\lambda(H)} = \sum_{n\geq 0}\frac{t^n\pi_\lambda(H^n)}{n!}.$$

So

$$\text{tr}\,\pi_\lambda(\exp tH) = \sum_{n\geq 0} \text{tr}\,\pi_\lambda(H^n)\frac{t^n}{n!}\,. \tag{11}$$

From Weyl's character formula (5)

$$\prod_{\alpha\in P}(e^{t\alpha(H)/2} - e^{-t\alpha(H)/2})\,\text{tr}\,\pi_\lambda(\exp tH) = \sum_{s\in W}\epsilon(s)e^{ts(\lambda+\delta)(H)}, \qquad t\in \mathbf{R},\ H\in\mathfrak{h}.$$

Using above formula in (11) we conclude

$$\prod_{\alpha\in P}(e^{t\alpha(H)/2} - e^{-t\alpha(H)/2})\sum_{n\geq 0}\frac{\text{tr}\,\pi_\lambda(H^n)t^n}{n!} = \sum_{s\in W}\epsilon(s)e^{ts(\lambda+\delta)(H)}. \tag{12}$$

Upon expanding $\Pi_{\alpha\in P}(e^{t\alpha(H)/2} - e^{-t\alpha(H)/2})$ and letting $d = [P]$ we get

$$\prod_{\alpha\in P}(e^{t\alpha(H)/2} - e^{-t\alpha(H)/2}) = t^d\pi(H)\sum_{m\geq 0}q'_m t^{2m}.$$

For sufficiently small $|t|$ above converges, hence from (12)

$$t^d\pi(H)\sum_{n\geq 0}\frac{\text{tr}\,\pi_\lambda(H^n)}{n!}t^n = \left[\sum_{m>0}q'_m t^{2m}\right]^{-1}\sum_{s\in W}\epsilon(s)e^{ts(\lambda+\delta)(H)}.$$

Since $q'_0 = 1$, for small $|t|$, we have $(\Sigma_{m\geq 0}q'_m t^{2m})^{-1} = \Sigma_{m\geq 0}q_m t^{2m}$. Using this in above formula, expanding the right-hand side, and equating the coefficients of t^{n+d}, we get

$$\frac{\pi(H)}{n!}\,\text{tr}\,\pi_\lambda(H^n) = \frac{\sum_{s\in W}\epsilon(s)(s(\lambda+\delta)(H))^{n+d}}{(n+d)!} + q_1\frac{\sum_{s\in W}\epsilon(s)(s(\lambda+\delta)(H))^{n+d-2}}{(n+d-2)!} + \cdots.$$

Combining this with (10) we get

$$\left(\prod_{\alpha\in P}\langle\delta,\alpha\rangle\right)^{-1}\frac{\pi(H)}{n!}\tilde{\omega}(\lambda+\delta)\beta_p(z_{H,n})(\lambda)$$
$$= \frac{\sum_{s\in W}\epsilon(s)(s(\lambda+\delta)(H))^{n+d}}{(n+d)!} + q_1\frac{\sum_{s\in W}\epsilon(s)(s(\lambda+\delta)(H))^{n+d-2}}{(n+d-2)!} + \cdots.$$

q_i's are independent of λ, and since above is true for $\lambda\in\mathfrak{D}_p^+$, it is true for $\lambda\in\mathfrak{h}_c^*$. In above formula we equate components which are homogeneous polynomials of degree $(n+d)$ in λ:

$$\frac{\pi(H)}{n!}\tilde{\omega}(\lambda)\beta_p(z_{H,n})(\lambda) = \prod_{\alpha\in P}\langle\lambda,\alpha\rangle\frac{\sum_{s\in W}\epsilon(s)(s\lambda(H))^{n+d}}{(n+d)!}.$$

Substituting above in

$$\int_G \exp B(H^x, H'_\mu)\, dx = \sum_{n \geq 0} \frac{1}{n!} \beta_p(z_{H,n})(\mu)$$

obtained earlier we conclude

$$(13)\qquad \pi(H)\tilde{\omega}(\mu) \int_G \exp B(H^x, H'_\mu)\, dx = \left(\prod_{\alpha \in p} \langle \delta, \alpha \rangle\right) \sum_{m \geq d} \frac{1}{m!} \sum_{s \in W} \epsilon(s)((s\lambda)(H))^m.$$

For $p \geq 0$ define $f_p(\lambda) = \Sigma_{s \in W} \epsilon(s)((s\lambda)(H))^p$, $\lambda \in \mathfrak{h}_c^*$. f_p is homogeneous of degree p and is skew symmetric. It follows [1] that $f_p = 0$, $p < d$. In view of above, we can rewrite (13) as

$$\pi(H)\tilde{\omega}(\mu) \int_G \exp B(H^x, H'_\mu)\, dx$$

$$= \left(\prod_{\alpha \in p} \langle \delta, \alpha \rangle\right) \sum_{m \geq 0} \frac{1}{m!} \sum_{s \in W} \epsilon(s)((s\mu)(H))^m = \left(\prod_{\alpha \in P} \langle \delta, \alpha \rangle\right) \sum_{s \in W} \epsilon(s) e^{(s\mu)(H)}.$$

Replacing H'_μ by H', and noting that $\tilde{\omega}(\mu) = \pi(H'_\mu)$ and $(s\mu)(H) = B(sH, H'_\mu)$ we obtain the result.

From this theorem we obtain immediately the sought-for formulae for the eigenfunctions $f_{X'}$.

Theorem 2.7. *Let $f_{X'}$ be defined as in Lemma 2.1, then*

$$(14)\qquad f_{H'}(H) = \left(\prod_{\alpha \in P} \langle \delta, \alpha \rangle\right) \frac{\sum_{s \in W} \epsilon(s) e^{B(sH, H')}}{\pi(H)\pi(H')} \qquad (H, H' \in \mathfrak{h}').$$

3. **The explicit calculation of $\delta'_{\mathfrak{h}}(\partial(p))$ $(p \in I_s(\mathfrak{g}_c))$.** We shall now prove (3), for the case G is compact. Let $\mathfrak{g}$ be of compact type and define

$$(15)\qquad \delta_{\mathfrak{h}}(D) = \pi \circ \delta'_{\mathfrak{h}}(D) \circ \pi^{-1} \qquad (D \in \mathrm{Diff}(U)).$$

Lemma 3.1. *Let $E \in \mathrm{Diff}(\mathfrak{h}')$ such that $Eg_{H'} = 0$ on $\mathfrak{h}'$ where $g_{H'}(H) = \Sigma_{s \in W} \epsilon(s) e^{B(sH, H')}$ $(H, H' \in \mathfrak{h}')$; then $E \equiv 0$.*

Proof. Choose $q_1, \cdots, q_k$ linearly independent in $S(\mathfrak{h}_c)$; then $E = \Sigma_{i=1}^k g_i \partial(q_i)$, $g_i \in C^\infty(\mathfrak{h}')$. $Eg_{H'} = 0$ implies

$$(16)\qquad \sum_{i=1}^k \sum_{s \in W} \epsilon(s) g_i(H) \tilde{q}_i(sH') e^{B(sH, H')} \equiv 0 \qquad (H, H' \in \mathfrak{h}')$$

Fix $H \in \mathfrak{h}'$, then the points sH, $s \in W$, are distinct (Chevalley). Thus $H' \to B(sH, H')$ are distinct linear forms. From a result of [3] and (16) we conclude

$$\sum_{i=1}^{k} \epsilon(s) g_i(H) \widetilde{q}_i^{\,s}(H') \equiv 0 \qquad (H' \in \mathfrak{h}).$$

Taking $s = 1$ and using the linear independence of q_i's we obtain $g_i(H) = 0$, $1 \leq i \leq k$. Therefore $E \equiv 0$.

Lemma 3.2. *For* $p \in I_s(\mathfrak{g}_c)$, $\delta_{\mathfrak{h}}(\partial(p)) = \partial(p_{\mathfrak{h}})$.

Proof. *If* g *is a function on* $(\mathfrak{h}')^G$, *let* $\overline{g} = g|_{\mathfrak{h}'}$. From (14) we get

$$\overline{f}_{H'}(H;\, \pi^{-1} \circ \partial(p_{\mathfrak{h}}) \circ \pi) = \widetilde{p}(H') \overline{f}_{H'}(H).$$

But Lemma 2.1 implies $\partial(p) f_{H'} = \widetilde{p}(H') f_{H'}$ on $\mathfrak{h}$. From the definition of radial component it follows that

$$\overline{f}_{H'}(H;\, \delta'_{\mathfrak{h}}(\partial(p))) = \widetilde{p}(H') \overline{f}_{H'}(H) \qquad (H \in \mathfrak{h}').$$

Above formulae along with (15) imply $(\partial(p_{\mathfrak{h}}) - \delta_{\mathfrak{h}}(\partial(p))) g_{H'} = 0$ on $\mathfrak{h}'$ where $g_{H'}(H) = \sum_{s \in W} \epsilon(s) e^{B(sH,\, H')}$ $(H, H' \in \mathfrak{h})$. The result follows from Lemma 3.1.

We are now in a position to prove (3) for an arbitrary semisimple Lie algebra over $\mathbf{R}$. Let $\mathfrak{g}$ be one such and G a connected Lie group with Lie algebra.

Lemma 3.3. *Let* $p \in I_s(\mathfrak{h}_c)$, *then* $\delta'_{\mathfrak{h}}(\partial(p)) = \pi^{-1} \circ \partial(p_{\mathfrak{h}}) \circ \pi$ *on* $\mathfrak{h}'$ *if and only if* $\partial(p_{\mathfrak{h}})(\pi q_{\mathfrak{h}}) = \pi(\partial(p) q)_{\mathfrak{h}}$ *on* $\mathfrak{h}_c$ *for all* $q \in I_p(\mathfrak{g}_c)$.

Proof. Let $E = \delta'_{\mathfrak{h}}(\partial(p)) - \pi^{-1} \circ \partial(p_{\mathfrak{h}}) \circ \pi$ and $\mathfrak{A}$ the algebra of all real-valued G-invariant polynomials on $\mathfrak{g}$. It can be shown(2) $E \equiv 0$ on $\mathfrak{h}'$ if and only if $E\overline{g} = 0$ for all $g \in \mathfrak{A}$. Recall that $\overline{g} = g|_{\mathfrak{h}'}$. This is equivalent to $\pi^{-1} \circ \partial(p_{\mathfrak{h}})(\pi q_{\mathfrak{h}}) = \delta'_{\mathfrak{h}}(\partial(p)) q_{\mathfrak{h}}$ on $\mathfrak{h}'$ for all $q \in I_p(\mathfrak{g}_c)$. Using the definition of $\delta'_{\mathfrak{h}}(\partial(p))$ and above, we get $\partial(p_{\mathfrak{h}})(\pi q_{\mathfrak{h}}) = \pi(\partial(p) q)_{\mathfrak{h}}$ on $\mathfrak{h}'$. Since we are dealing with polynomials this implies the result.

Theorem 3.4. *Let* $\mathfrak{g}$ *be a real semisimple Lie algebra,* $\mathfrak{g}_c$ *its complexification. Let* $p \in I_s(\mathfrak{g}_c)$; *then* $\delta_{\mathfrak{h}}(\partial(p)) = \partial(p_{\mathfrak{h}})$.

Proof. Let U be a compact real form of $\mathfrak{g}_c$, and $\mathfrak{b}$ a Cartan subalgebra of it. Then $\mathfrak{b}_c = \mathbf{C} \cdot \mathfrak{b}$ is a Cartan subalgebra of $\mathfrak{g}_c$ and hence $\mathfrak{b}_c^x = \mathfrak{h}_c$ for some $x \in G_c$.

(2) It suffices to show $E_{H_0} = 0$ for $H_0 \in \mathfrak{h}'$. $I_p(\mathfrak{g}_c)$ is finitely generated by l algebraically independent homogeneous elements of positive degree. It can be shown that we can choose these generators so that their restrictions to $\mathfrak{g}$ are real-valued. Let $\{q_1, \cdots, q_l\}$ be one such set. Then by the Chevalley theorem $\{\overline{q}_1, \cdots, \overline{q}_l\}$ generates $I_p(\mathfrak{h}_c)$. A theorem of Shephard and Todd [5] implies $\{\overline{q}_1, \cdots, \overline{q}_l\}$ forms a C^∞ coordinate system around H_0. A straightforward argument now implies the claim.

Let π be the product of positive roots of $(\mathfrak{g}_c, \mathfrak{b}_c)$. We arrange matters so that $\pi = \pi_{\mathfrak{b}}^x$. By Lemma 3.2, $\delta_{\mathfrak{b}}'(\partial(p)) = \pi^{-1} \circ \partial(p_{\mathfrak{b}}) \circ \pi$ on $\mathfrak{b}'$. By Lemma 3.3, $\partial(p_{\mathfrak{b}})(\pi_{\mathfrak{b}} q_{\mathfrak{b}}) = \pi_{\mathfrak{b}}(\partial(p)q)_{\mathfrak{b}}$ on $\mathfrak{b}_c$. We then have

$$\partial(p_{\mathfrak{b}}^x)(\pi_{\mathfrak{b}}^x q_{\mathfrak{b}}^x) = \pi_{\mathfrak{b}}^x((\partial(p)q)_{\mathfrak{b}})^x.$$

But $p_{\mathfrak{b}}^x = p_{\mathfrak{h}}$, $q_{\mathfrak{b}}^x = q_{\mathfrak{h}}$. So we have $\partial(p_{\mathfrak{h}})(\pi q_{\mathfrak{h}}) = \pi(\partial(p)q)_{\mathfrak{h}}$. This proves the theorem in view of Lemma 3.3.

4. **A theorem on $\delta_{\mathfrak{h}}'(D)$ when $D \in \mathfrak{D}(\mathfrak{g}_c)$.** We now use the previous results to obtain the main theorem concerning $\delta_{\mathfrak{h}}'(D)$, $D \in \mathfrak{D}(\mathfrak{g}_c)$ (cf. Harish-Chandra [2]). For any $p \in S(\mathfrak{g}_c)$ define $\mu_p(D) = \partial(p) \circ D - D \circ \partial(p)$ $(D \in \text{Diff}((\mathfrak{h}')^G))$. μ_p is a derivation of $\mathfrak{D}(\mathfrak{g}_c)$. Define $\mu_p^m(D) = \mu(\mu_p^{m-1}(D))$, $m \geq 2$.

Lemma 4.1. *Let $D \in \mathfrak{D}(\mathfrak{g}_c)$, $p \in S(\mathfrak{g}_c)$. Then there is an integer $m = m(p, D) \geq 0$ such that $\mu_p^m(D) = 0$.*

Proof. Since we can write $D = \sum_{i=1}^s p_i \partial(q_i)$, $p_i \in P(\mathfrak{g}_c)$, $q_i \in S(\mathfrak{g}_c)$ we may assume $D = p_1\partial(q_1)$. We assert $\mu_p(D)$ can be written as $\sum_{i=1}^k p_i' \partial(q_i')$ with $\deg p_i' < \deg p_1$ for all i. Write $E = F\partial(q_1)$ where $F = [\partial(p), p_1]$. Direct calculations show that F can be written as $\sum_{j=1}^N f_j \partial(g_j)$ where each f_j is of the form $\partial(h_j)p_1$ for some $h_j \in P(\mathfrak{g}_c)$ which is homogeneous of positive degree. So $\deg f_j < \deg p_1$ for all j. Since $E = \sum f_j \partial(g_j q_1)$ the result follows by induction on $\max_i \deg p_i$.

Lemma 4.2. *Let $U \subset \mathfrak{h}$ can be a connected and open set, $f \in C^\infty(U)$. Suppose for every $p \in I_s^+(\mathfrak{h}_c)$, there is an integer $m = m(p) > 0$ such that $\partial(p)^m f = 0$ on U. Then f is the restriction to U of a polynomial on $\mathfrak{h}_c$.*

Proof. Let $p_1, \cdots, p_l$ be homogeneous generators of $I_s(\mathfrak{h}_c)$. Then it can be seen that the assumption implies $\partial(p_i)^m f = 0$ for some $m \geq 0$, $1 \leq i \leq l$. Let $d_i = \deg(p_i)$ and $k = m(d_1 + \cdots + d_l)$. For $H \in \mathfrak{h}$ consider $Q(\zeta) = \Pi_{s \in W}(\zeta^k - (sH)^k)$. $Q(H) = 0$, so $\partial(Q(H))f = 0$. Let $\omega = [W]$. Then

$$Q(\zeta) = (\zeta^k)^\omega + (\zeta^k)^{\omega-1} Q_{1,H} + \cdots + Q_{\omega,H}$$

where $Q_{j,H}$ is a homogeneous element of $I_s(\mathfrak{h}_c)$ of degree jk. For fixed j, $Q_{j,H}$ is a linear combination of monomials $p_1^{a_1} \cdots p_l^{a_l}$ with $d_1 a_1 + \cdots + d_l a_l = jk \geq k = m(d_1 + \cdots + d_l)$. Hence there is at least one $a_i > m$, so that $\partial(p_1^{a_1} \cdots p_l^{a_l})f = 0$. Thus $\partial(Q_{j,H})f = 0$. Hence $\partial(H^{k\omega})f = 0$. As U is connected, this implies f is a polynomial of degree $\leq lk\omega$.

Lemma 4.3. *Let $U \subset \mathfrak{h}$ be a connected open set, $E \in \text{Diff}(U)$. Suppose for each $p \in I_s^+(\mathfrak{h}_c)$ there is an integer $m = m(p) > 0$ such that $\mu_p^m(E) = 0$. Then there* *is an* $F \in \mathfrak{D}(\mathfrak{h})$ *such that* $E = F$ *on* U, *i.e.* E *has polynomial coefficients.*

Proof. Write $E = \Sigma_{i=1}^N f_i \partial(g_i)$, $f_i \in C^\infty(U)$, $g_i \in S(\mathfrak{h}_c)$. We will show $f_i \in P(\mathfrak{h}_c)$. We may assume g_i are linearly independent, homogeneous and $\deg g_1 \le \deg g_2 \le \cdots \le \deg g_N$. Also we may assume $f_i \notin P(\mathfrak{h}_c)$. Let $d = \min_i \deg g_i$ and $s \ge 1$ such that $\deg g_i = d$, $1 \le i \le s$, $\deg g_i > d$, $s < i \le N$. Let $p \in I_s(\mathfrak{h}_c)$. It can be shown

$$\mu_p^m(E) = \sum_{i=1}^{N} \sum_{r=0}^{m} (-1)^r \binom{m}{r} (\partial(p))^{m-r}(f_i\partial(g_i))\partial(p)^r. \tag{17}$$

Let M_f be the operator of multiplication by f, then $\partial(p)^{m-r}M_{f_i} = \Sigma_t U_{i,t}\partial(V_{i,t})$ where $U_{i,t} \in C^\infty(U)$ and $V_{i,t}$ homogeneous. We then get

$$\partial(p)^{m-r}(f_i\partial(g_i))\partial(p)^r = \sum_t U_{i,t}\partial(V_{i,t}g_ip^r).$$

If $r = 0$ we can write $\partial(p)^m M_{f_i} = M_{\partial(p)^m f_i} + \Sigma_t U_{i,t}\partial(V_{i,t})$ where the $V_{i,t}$ are homogeneous elements of strictly positive degree. We then have

$$\partial(p)^m(f_i\partial(g_i)) = \sum_t U_{i,t}\partial(V_{i,t}g_i).$$

It follows from the above two equations that

$$\mu_p^m(E) = \sum_{i=1}^{s} (\partial(p)^m f_i)\partial(g_i) + \sum_{j=1}^{Nn} f_{m_j}\partial(g_{m_j})$$

where $f_{m_j} \in C^\infty(U)$ and g_{m_j} are homogeneous elements of $S(\mathfrak{h}_c)$ of degree $> d$. If we choose m so that $\mu_p^m(E) = 0$, then we conclude from above formula that $\partial(p)^m f_i = 0$, $1 \le i \le s$. Lemma 4.2 now implies that $f_i \in P(\mathfrak{h}_c)$, a contradiction.

Theorem 4.4. *Let* $D \in \mathfrak{I}(\mathfrak{g})$ *and* $\delta_{\mathfrak{h}}(D) = \pi \circ \delta'_{\mathfrak{h}}(D) \circ \pi^{-1}$. *There is a unique element of* $\mathfrak{D}(\mathfrak{h})$ *which coincides with* $\delta_{\mathfrak{h}}(D)$ *on* $\mathfrak{h}'$.

Proof. We need only to prove the existence. By Theorem 2.1, $\delta_{\mathfrak{h}}: D' \longrightarrow \pi \circ \delta'_{\mathfrak{h}}(D') \circ \pi^{-1}$ is a homomorphism of $\mathfrak{I}((\mathfrak{h}')^G)$ into Diff$(\mathfrak{h}')$. Suppose now $p \in I_s(\mathfrak{g}_c)$. By Lemma 4.1, there is an integer $m \ge 0$ such that $\mu_p^m(D) = 0$. So

$$\sum_{r=0}^{m} (-1)^r \binom{m}{r} \partial(p)^{m-r} D \partial(p)^r = 0.$$

Apply homomorphism $\delta_{\mathfrak{h}}$ to above and note that by Theorem 3.4, $\delta_{\mathfrak{h}}(\partial(q)) = \partial(q_{\mathfrak{h}})$ $(q \in I_s(\mathfrak{g}_c))$ we will get

$$\sum_{r=0}^{m} (-1)^r \binom{m}{r} \partial(p_{\mathfrak{h}})^{m-r} \delta_{\mathfrak{h}}(D) \partial(p_{\mathfrak{h}})^r = 0 \quad \text{on } \mathfrak{h}'.$$

Comparing above with (17) we get

$$\mu^m_{p_{\mathfrak{h}}}(\delta_{\mathfrak{h}}(D)) = 0 \quad \text{on } \mathfrak{h}'.$$

Let U be a connected open set in $\mathfrak{h}'$. Since $p \longrightarrow p_{\mathfrak{h}}$ is an isomorphism of $I_s(\mathfrak{g}_c)$ onto $I_s(\mathfrak{h}_c)$, we conclude from above that given any $q \in I_s(\mathfrak{h}_c)$ which is homogeneous of positive degree, there is an integer $m = m(p) > 0$ such that $\mu^m_q(\delta_{\mathfrak{h}}(D)) = 0$ on U. By Lemma 4.3, we can find an $L \in \mathfrak{D}(\mathfrak{h})$ such that

(18) $$L = \delta_{\mathfrak{h}}(D) \quad \text{on } U.$$

It can be shown [1] that $\delta'_{\mathfrak{h}}(D)$ can be written as $\pi^{-N'} \circ F$ for some integer $N' \geq 0$ and $F \in \mathfrak{D}(\mathfrak{h})$. So we can write $\delta_{\mathfrak{h}}(D)$ as $\pi^{-N} \circ F$ for some $N \geq 0$ and $F \in \mathfrak{D}(\mathfrak{h})$. Comparing this with (18) and noting a rational function on $\mathfrak{h}_c$ is determined by its restriction on U, we conclude that $L = \delta_{\mathfrak{h}}(D)$ on $\mathfrak{h}'$.

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